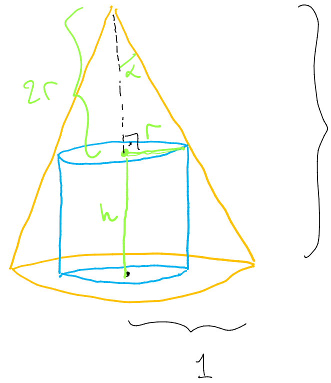
Section 4.8

Monday, March 30, 2020 3:21 PM

Example: Inside a box in the shape of a right circular cone with radius 1 m and height 2m will be placed a box in the shape of a right circular cylinder in such a way that the bases of the cone and the cylinder are on the same plane. What is the largest possible volume of the cylindrical box?

Solution:



The volume of the cylindrical box is

$$\pi r^2 h = \pi r^2 (2-2r)$$

$\boxed{\tan\alpha = \frac{1}{2} \qquad h + 2r = 2}$

So we wish to find the absolute maximum of the function

$f(r) = \pi r^2 (2-2r)$ over the interval $(0,1)$

$= 2\pi r^2 - 2\pi r^3$

$f'(r) = 4\pi r - 6\pi r^2 = \pi r(4-6r) = 0 \Rightarrow r = \frac{2}{3}$ is the only critical point.

r	0		$\frac{2}{3}$		1
f'		+	0	−	
f		↗		↘	

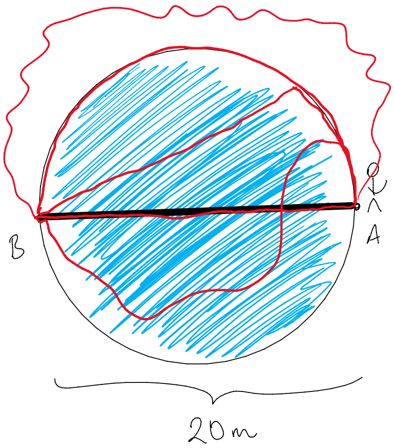
As $f'(r) > 0$ on $(0, \frac{2}{3})$ and f is cont, $f(\frac{2}{3}) \geq f(x)$ for all $0 < x < \frac{2}{3}$

— $f'(r) < 0$ — $(\frac{2}{3}, 1)$ ——, ——

Thus f has its abs. max on $(0,1)$ at $r = \frac{2}{3}$.

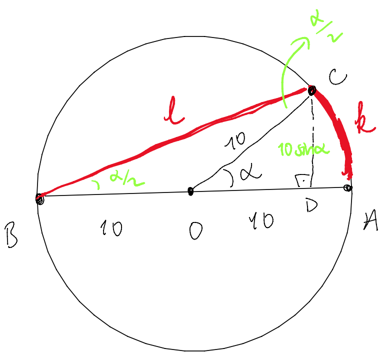
So the largest possible volume of this cylindrical box is $f(\frac{2}{3}) = 2\pi \frac{4}{9} - 2\pi \frac{8}{27}$ m^3.

Example:



A man is standing on the edge of a swimming pool in the shape of a circle at point A, as shown in the figure. He wants to go to the point B as quickly as possible, where $|AB|$ is the diameter of the pool. Given that this man can run $2\,m/s$ and can swim $1\,m/s$, what route should he take?

Solution:



The total travel time will be

$$\frac{k}{2} + \frac{l}{1} = \frac{\left(\cancel{2\pi}\cdot 10 \frac{\alpha}{\cancel{2\pi}}\right)}{2} + 20\cos\left(\frac{\alpha}{2}\right)$$

$|CD| = 10\sin\alpha$

$\frac{10\sin\alpha}{l} = \sin\left(\frac{\alpha}{2}\right) \Rightarrow \frac{10\cdot 2\cancel{\sin\frac{\alpha}{2}}\cos\frac{\alpha}{2}}{l} = \cancel{\sin\frac{\alpha}{2}} \Rightarrow l = 20\cos\frac{\alpha}{2}$

So we wish to minimize $f(\alpha) = 5\alpha + 20\cos\left(\frac{\alpha}{2}\right)$ over the interval $[0, \pi]$.

$f'(\alpha) = 5 + 20\cdot\left(-\sin\left(\frac{\alpha}{2}\right)\right)\cdot\frac{1}{2} = 5 - 10\sin\left(\frac{\alpha}{2}\right) = 0 \Rightarrow \sin\left(\frac{\alpha}{2}\right) = \frac{1}{2} \Rightarrow \alpha = \frac{\pi}{3}$ is the only critical point in $[0, \pi]$.

$f(0) = 20$

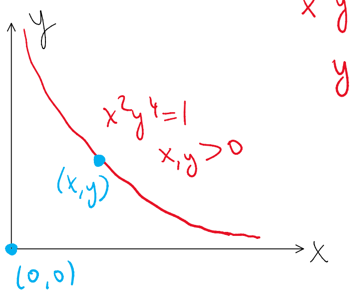
$f\left(\frac{\pi}{3}\right) = \frac{5\pi}{3} + 20\cos\left(\frac{\pi}{6}\right) = \frac{5\pi}{3} + 20\cdot\frac{\sqrt{3}}{2} = \frac{5\pi}{3} + 10\sqrt{3} > 22$

$f(\pi) = 5\pi < 20$

So in order to minimize the travel time, this man should run all the way over the edge.

Example: Find the shortest distance from the origin to the curve $x^2y^4 = 1$, $x, y > 0$.

Solution:



$x^2y^4 = 1 \overset{x,y>0}{\Rightarrow} y = \frac{1}{\sqrt{x}}$

The distance of the point (x,y) on this curve to the origin is

$$d = \sqrt{(x-0)^2 + (y-0)^2} = \sqrt{x^2+y^2} = \sqrt{x^2 + \frac{1}{x}}$$

We wish to find the absolute minimum of $f(x) = \sqrt{x^2 + \frac{1}{x}}$ over the interval $(0, \infty)$.

We can do this, just like before, OR we can try to find the point which maximizes $(f(x))^2 = \left(\sqrt{x^2+\frac{1}{x}}\right)^2 = x^2 + \frac{1}{x}$ over $(0, \infty)$. (Because $f(x) > 0$ on this interval.)

$g(x) = x^2 + \frac{1}{x}$ $\quad g'(x) = 2x - \frac{1}{x^2} = 0 \Rightarrow 2x = \frac{1}{x^2} \Rightarrow x^3 = \frac{1}{2}$

$x = \frac{1}{\sqrt[3]{2}}$

Exercise!

x	0		$\frac{1}{\sqrt[3]{2}}$		∞
g'		$-$	0	$+$	
g		$\searrow$		$\nearrow$	

As before, we can argue that g has its absolute minimum at $x = \frac{1}{\sqrt[3]{2}}$.

So the shortest distance of the origin to this curve is

$$f\left(\frac{1}{\sqrt[3]{2}}\right) = \sqrt{\frac{1}{\sqrt[3]{4}} + \sqrt[3]{2}}$$